

On the absence of Volterra correct restrictions and extensions of the Laplace operator

Bazarkan N. Biyarov

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Abstract

At the beginning of the last century J. Hadamard constructed the well-known example illustrating the incorrectness of the Cauchy problem for elliptic-type equations. If the Cauchy problem for some differential equation is correct, then it is usually a Volterra problem, i.e., the inverse operator is a Volterra operator. At present, not a single Volterra correct restriction or extension for elliptic-type equations is known. In the present paper, we prove the absence of Volterra correct restrictions of the maximal operator \widehat{L} and Volterra correct extensions of the minimal operator L_0 generated by the Laplace operator in $L_2(\Omega)$, where Ω is the unit disk.

1 Introduction

Let us present some definitions, notation, and terminology.

In a Hilbert space H , we consider a linear operator L with domain $D(L)$ and range $R(L)$. By the *kernel* of the operator L we mean the set

$$\text{Ker } L = \{f \in D(L) : Lf = 0\}.$$

Definition 1.1. An operator L is called a *restriction* of an operator L_1 , and L_1 is called an *extension* of an operator L , briefly $L \subset L_1$, if:

- 1) $D(L) \subset D(L_1)$,
- 2) $Lf = L_1f$ for all f from $D(L)$.

Definition 1.2. A linear closed operator L_0 in a Hilbert space H is called *minimal* if $\overline{R(L_0)} \neq H$ and there exists a bounded inverse operator L_0^{-1} on $R(L_0)$.

Definition 1.3. A linear closed operator \widehat{L} in a Hilbert space H is called *maximal* if $R(\widehat{L}) = H$ and $\text{Ker } \widehat{L} \neq \{0\}$.

Definition 1.4. A linear closed operator L in a Hilbert space H is called *correct* if there exists a bounded inverse operator L^{-1} defined on all of H .

Definition 1.5. We say that a correct operator L in a Hilbert space H is a *correct extension* of minimal operator L_0 (*correct restriction* of maximal operator \widehat{L}) if $L_0 \subset L$ ($L \subset \widehat{L}$).

Definition 1.6. We say that a correct operator L in a Hilbert space H is a *boundary correct extension* of a minimal operator L_0 with respect to a maximal operator \widehat{L} if L is simultaneously a correct restriction of the maximal operator \widehat{L} and a correct extension of the minimal operator L_0 , that is, $L_0 \subset L \subset \widehat{L}$.

Let \widehat{L} be a maximal linear operator in a Hilbert space H , let L be any known correct restriction of \widehat{L} , and let K be an arbitrary linear bounded (in H) operator satisfying the following condition:

$$R(K) \subset \text{Ker } \widehat{L}. \quad (1.1)$$

Then the operator L_K^{-1} defined by the formula (see [1])

$$L_K^{-1}f = L^{-1}f + Kf, \quad (1.2)$$

describes the inverse operators to all possible correct restrictions L_K of \widehat{L} , i.e., $L_K \subset \widehat{L}$.

Let L_0 be a minimal operator in a Hilbert space H , let L be any known correct extension of L_0 , and let K be a linear bounded operator in H satisfying the conditions

- a) $R(L_0) \subset \text{Ker } K$,
- b) $\text{Ker}(L^{-1} + K) = \{0\}$,

then the operator L_K^{-1} defined by formula (1.2) describes the inverse operators to all possible correct extensions L_K of L_0 (see [1]).

Let L be any known boundary correct extension of L_0 , i.e., $L_0 \subset L \subset \widehat{L}$. The existence of at least one boundary correct extension L was proved by Vishik in [2]. Let K be a linear bounded (in H) operator satisfying the conditions

- a) $R(L_0) \subset \text{Ker } K$,
- b) $R(K) \subset \text{Ker } \widehat{L}$,

then the operator L_K^{-1} defined by formula (1.2) describes the inverse operators to all possible boundary correct extensions L_K of L_0 (see [1]).

Definition 1.7. A bounded operator A in a Hilbert space H is called *quasinilpotent* if its spectral radius is zero, that is, the spectrum consists of the single point zero.

Definition 1.8. An operator A in a Hilbert space H is called a *Volterra operator* if A is compact and quasinilpotent.

Definition 1.9. A correct restriction L of a maximal operator \widehat{L} ($L \subset \widehat{L}$), a correct extension L of a minimal operator L_0 ($L_0 \subset L$) or a boundary correct extension L of a minimal operator L_0 with respect to a maximal operator \widehat{L} ($L_0 \subset L \subset \widehat{L}$), will be called *Volterra* if the inverse operator L^{-1} is a Volterra operator.

We denote by $\mathfrak{S}_\infty(H, H_1)$ the set of all linear compact operators acting from a Hilbert space H to a Hilbert space H_1 . If $T \in \mathfrak{S}_\infty(H, H_1)$, then T^*T is a non-negative self-adjoint operator in $\mathfrak{S}_\infty(H) \equiv \mathfrak{S}_\infty(H, H)$ and, moreover, there is a non-negative unique self-adjoint root $|T| = (T^*T)^{1/2}$ in $\mathfrak{S}_\infty(H)$. The eigenvalues $\lambda_n(|T|)$ numbered, taking into account their multiplicity, form a monotonically converging to zero sequence of non-negative numbers. These numbers are usually called *s-numbers* of the operator T and denoted by $s_n(T)$, $n \in \mathbb{N}$. We denote by $\mathfrak{S}_p(H, H_1)$ the set of all compact operators $T \in \mathfrak{S}_\infty(H, H_1)$, for which

$$|T|_p^p = \sum_{j=1}^{\infty} s_j^p(T) < \infty, \quad 0 < p < \infty.$$

Obviously, if $\text{rank } |T| = r < \infty$, then $s_n(T) = 0$, for $n = r + 1, r + 2, \dots$. Operators of finite rank certainly belong to the classes $\mathfrak{S}_p(H, H_1)$ for all $p > 0$.

In the Hilbert space $L_2(\Omega)$, where Ω is the unit disk in \mathbb{R}^2 with boundary $\partial\Omega$, let us consider the minimal L_0 and maximal \widehat{L} operators generated by the Laplace operator

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \quad (1.3)$$

The closure L_0 in the space $L_2(\Omega)$ of the Laplace operator (1.3) with the domain $C_0^\infty(\Omega)$ is the *minimal operator corresponding to the Laplace operator*.

The operator \widehat{L} , adjoint to the minimal operator L_0 corresponding to the Laplace operator is the *maximal operator corresponding to the Laplace operator* (see [3]). Note that

$$D(\widehat{L}) = \{u \in L_2(\Omega) : \widehat{L}u = -\Delta u \in L_2(\Omega)\}.$$

Denote by L_D the operator, corresponding to the Dirichlet problem with the domain

$$D(L_D) = \{u \in W_2^2(\Omega) : u|_{\partial\Omega} = 0\}.$$

Then, by virtue of (1.2), the inverse operators L^{-1} to all possible correct restrictions of the maximal operator \widehat{L} corresponding to the Laplace operator (1.3) have the following form:

$$u \equiv L^{-1}f = L_D^{-1}f + Kf, \quad (1.4)$$

where, by virtue of (1.1), K is an arbitrary linear operator bounded in $L_2(\Omega)$ with

$$R(K) \subset \text{Ker } \widehat{L} = \{u \in L_2(\Omega) : -\Delta u = 0\}.$$

Then the direct operator L is determined from the following problem:

$$\widehat{L}u \equiv -\Delta u = f, \quad f \in L_2(\Omega), \quad (1.5)$$

$$D(L) = \{u \in D(\widehat{L}) : [(I - K\widehat{L})u]|_{\partial\Omega} = 0\}, \quad (1.6)$$

where I is the unit operator in $L_2(\Omega)$. There are no other linear correct restrictions of the operator \widehat{L} (see [4]).

The operators $(L^*)^{-1}$, corresponding to the adjoint operators L^*

$$v \equiv (L^*)^{-1}g = L_D^{-1}g + K^*g,$$

describe the inverse operators to all possible correct extensions of the minimal operator L_0 if and only if K satisfies the condition (see [4]):

$$\text{Ker}(L_D^{-1} + K^*) = \{0\}.$$

Note that the last condition is equivalent to the following: $\overline{D(L)} = L_2(\Omega)$. If the operator K from (1.4) satisfies one more additional condition

$$KR(L_0) = \{0\},$$

then the operator L corresponding to problem (1.5), (1.6), will turn out to be a boundary correct extension.

Now we state the main result.

2 Main results

We pass to the polar coordinate system:

$$x = r \cos \varphi, \quad y = r \sin \varphi.$$

Then the operator

$$\widehat{L}u \equiv -\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = -u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\varphi\varphi} = f(r, \varphi), \quad (2.1)$$

on

$$D(\widehat{L}) = \{u \in L_2(\Omega) : \Delta u \in L_2(\Omega)\},$$

is the maximal operator (see [5]). Any correct restriction L acts as the maximal operator \widehat{L} on the domain

$$D(L) = \{u \in D(\widehat{L}) : [(I - K\widehat{L})u]|_{r=1} = 0\}, \quad (2.2)$$

where K is any bounded linear operator in $L_2(\Omega)$ that $R(K) \subset \text{Ker } \widehat{L}$. To be Volterra of L is necessary compactness of L^{-1} . From (1.4) note that L^{-1} is compact if and only if K is a compact operator. Then for K the Schmidt expansion takes place (see [6, p. 47(28)])

$$K = \sum_{j=1}^{\infty} s_j(\cdot, Q_j) F_j \quad (2.3)$$

where $\{Q_j\}_1^{\infty}$ is orthonormal system in $L_2(\Omega)$, $\{F_j\}_1^{\infty}$ is orthonormal system in $\text{Ker } \widehat{L}$ and $\{s_j\}_1^{\infty}$ is a monotone sequence of non-negative numbers converging to zero. The series on the right side of (2.3) converges in the uniform operator norm. We now state the main result of this paper.

Theorem 2.1. *Let \widehat{L} be a maximal operator generated by the Laplace (1.3) in $L_2(\Omega)$. Then any correct restriction L of the maximal operator \widehat{L} , i.e., the problem (2.1) and (2.2) cannot be Volterra.*

Proof. Let us prove by contradiction. Suppose that there exists a Volterra correct restriction L . This is equivalent to the existence of a such compact operator K that the operator L has no non-zero eigenvalue. The general solution of the equation

$$\widehat{L}u = -\Delta u = -u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\varphi\varphi} = \lambda^2 u,$$

from the space $L_2(\Omega)$ has the form (see [7])

$$u(r, \varphi) = u_0(r, \varphi) - \int_0^r u_0(\rho, \varphi) \frac{\partial}{\partial \rho} J_0(\lambda \sqrt{r(r-\rho)}) d\rho,$$

where λ is any complex number, $u_0(r, \varphi)$ is the solution of the equation

$$\widehat{L}u_0 \equiv -\Delta u_0 = -u_{0rr} - \frac{1}{r}u_{0r} - \frac{1}{r^2}u_{0\varphi\varphi} = 0,$$

which is a harmonic function from the space $L_2(\Omega)$ and

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$$

is the Bessel function. Then, by virtue of (2.2) we obtain the equation

$$\begin{aligned} & u_0(1, \varphi) - \int_0^1 u_0(\rho, \varphi) \frac{\partial}{\partial \rho} J_0(\lambda \sqrt{1-\rho}) d\rho \\ & - \lambda^2 \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 u_0(\rho, \theta) \cdot \overline{Q_j(\rho, \theta)} \rho d\rho d\theta \\ & + \lambda^2 \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \overline{Q_j(\rho, \theta)} \cdot \int_0^\rho u_0(\tau, \theta) \frac{\partial}{\partial \tau} J_0(\lambda \sqrt{\rho(\rho-\tau)}) d\tau \rho d\rho d\theta = 0. \end{aligned} \quad (2.4)$$

The considered problem on the spectrum of the Laplace operator has no eigenvalues if and only if the equation (2.4) has no zeros as a function of λ . The harmonic function $u_0(\rho, \varphi)$ does not depend on λ . It is easy to notice that the left side of the equation is an entire function no higher than the first order. Then by virtue of Picard's theorem (see [8, p.264, 266]) this function have the form $Ce^{d\lambda}$, where $C(\varphi)$ and $d(\varphi)$ are a functions which are independent of λ . If you notice that the left side of the equation (2.4) is even with respect to the sign of λ , then $d = 0$. Equating these functions when $\lambda = 0$ we have $C = u_0(1, \varphi)$. Then we get the following

$$\begin{aligned} & - \int_0^1 u_0(\rho, \varphi) \frac{\partial}{\partial \rho} J_0(\lambda \sqrt{1-\rho}) d\rho \\ & - \lambda^2 \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 u_0(\rho, \theta) \cdot \overline{Q_j(\rho, \theta)} \rho d\rho d\theta \\ & + \lambda^2 \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \overline{Q_j(\rho, \theta)} \int_0^\rho u_0(\tau, \theta) \frac{\partial}{\partial \tau} J_0(\lambda \sqrt{\rho(\rho-\tau)}) d\tau \rho d\rho d\theta = 0. \end{aligned} \quad (2.5)$$

Divide both sides of (2.5) by λ^2 and let λ tend to zero. Then

$$\sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 u_0(\rho, \theta) \overline{Q_j(\rho, \theta)} \rho d\rho d\theta = -\frac{1}{4} \int_0^1 u_0(\rho, \varphi) d\rho. \quad (2.6)$$

Under the condition that (2.6) is fulfilled we obtain

$$\begin{aligned} & - \int_0^1 u_0(\rho, \varphi) \left[\frac{\partial J_0}{\partial \rho}(\lambda \sqrt{1-\rho}) + \frac{1}{4} \right] d\rho \\ & + \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \overline{Q_j(\rho, \theta)} \int_0^\rho u_0(\tau, \theta) \frac{\partial}{\partial \tau} J_0(\lambda \sqrt{\rho(\rho-\tau)}) d\tau \rho d\rho d\theta = 0. \end{aligned} \quad (2.7)$$

On the left side of the equation (2.7) we make the change of variables: in the first summand $t = \sqrt{1-\rho}$, in the second summand $t = \sqrt{\rho(\rho-\tau)}$. Then we have

$$\begin{aligned} & \int_0^1 u_0(1-t^2, \varphi) \left[\frac{J'_0(\lambda t)}{2\lambda t} + \frac{1}{4} \right] 2t dt \\ & - \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 J'_0(\lambda t) \int_t^1 u_0\left(\frac{\rho^2-t^2}{\rho}, \theta\right) \overline{Q_j(\rho, \theta)} \rho d\rho dt d\theta = 0. \end{aligned} \quad (2.8)$$

For the Bessel function has the following equalities

$$\frac{J'_0(\lambda t)}{2\lambda t} + \frac{1}{4} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{((n+1)!)^2} \left(\frac{\lambda t}{2}\right)^{2n} \cdot (n+1),$$

and

$$\lambda J'_0(\lambda t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\lambda t}{2}\right)^{2n} \cdot n \cdot \frac{2}{t}.$$

Substitute them into (2.8) and equate the coefficients of λ^{2n} to zero

$$\begin{aligned} & \int_0^1 u_0(1-t^2, \varphi) \frac{-1}{4(n+1)} \cdot t^{2n} \cdot 2t dt \\ & - \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 n t^{2n} \cdot \frac{2}{t} \int_t^1 u_0\left(\frac{\rho^2-t^2}{\rho}, \theta\right) \overline{Q_j(\rho, \theta)} \rho d\rho dt d\theta = 0. \end{aligned}$$

We do the conversion of the following form

$$\frac{1}{n+1} \cdot t^{2n} = \frac{2}{t^2} \int_0^t \tau^{2n+1} d\tau, \quad n \cdot t^{2n} = \frac{t}{2} \cdot \frac{\partial}{\partial t} (t^{2n}).$$

Then

$$\begin{aligned} & \int_0^1 t^{2n} \cdot \left\{ t \int_t^1 u_0(1-\tau^2, \varphi) \frac{d\tau}{\tau} \right. \\ & \left. - \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial t} \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \int_t^1 u_0\left(\frac{\rho^2-t^2}{\rho}, \theta\right) \overline{Q_j(\rho, \theta)} \rho d\rho d\theta \right\} dt = 0. \end{aligned}$$

In view of the completeness of the system of functions $\{t^{2n}\}_1^{\infty}$ in $L_2(0, 1)$ we obtain (see [9, p. 107])

$$t \int_t^1 u_0(1-\tau^2, \varphi) \frac{d\tau}{\tau} - \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial t} \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \int_t^1 u_0\left(\frac{\rho^2-t^2}{\rho}, \theta\right) \overline{Q_j(\rho, \theta)} \rho d\rho d\theta = 0.$$

Integrating this equation from t to 1, we get

$$\begin{aligned} & \int_t^1 u_0(1-\tau^2, \varphi) \frac{\tau^2-t^2}{2\tau} d\tau \\ & + \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 u_0\left(\frac{\rho^2-t^2}{\rho}, \theta\right) \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \overline{Q_j(\rho, \theta)} \rho d\rho d\theta = 0. \end{aligned} \tag{2.9}$$

where $0 \leq t \leq 1$, $0 \leq \varphi < 2\pi$. Note that the condition (2.9) contains the condition (2.6) as a particular case when $t = 0$. Condition (2.9) will turn out to be the Volterra criterion of the correct restriction L , if it holds for any harmonic function $u_0(r, \varphi)$ from $L_2(\Omega)$.

By Poisson's formula

$$u_0(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r \cos(\varphi-\gamma) + r^2} u_0(1, \gamma) d\gamma,$$

the equality (2.9) is transformed to

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} u_0(1, \gamma) \left\{ \int_t^1 \frac{1-(1-\tau^2)^2}{1-2(1-\tau^2) \cos(\varphi-\gamma) + (1-\tau^2)^2} \cdot \frac{\tau^2-t^2}{2\tau} d\tau \right. \\ & \left. + \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 \frac{1-\left(\frac{\rho^2-t^2}{\rho}\right)^2}{1-2\left(\frac{\rho^2-t^2}{\rho}\right) \cos(\theta-\gamma) + \left(\frac{\rho^2-t^2}{\rho}\right)^2} \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \overline{Q_j(\rho, \theta)} \rho d\rho d\theta \right\} d\gamma = 0. \end{aligned}$$

Considering the density of the set of functions $u_0(1, \varphi)$ in $L_2(0, 2\pi)$ for almost all values of t ($0 \leq t \leq 1$), φ ($0 \leq \varphi < 2\pi$) we obtain the equality

$$\begin{aligned} & \int_t^1 \frac{1 - (1 - \tau^2)^2}{1 - 2(1 - \tau^2) \cos(\varphi - \gamma) + (1 - \tau^2)^2} \cdot \frac{\tau^2 - t^2}{2\tau} d\tau \\ & + \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 \frac{1 - \left(\frac{\rho^2 - t^2}{\rho}\right)^2}{1 - 2\left(\frac{\rho^2 - t^2}{\rho}\right) \cos(\theta - \gamma) + \left(\frac{\rho^2 - t^2}{\rho}\right)^2} \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \overline{Q_j(\rho, \theta)} \rho d\rho d\theta = 0. \end{aligned} \quad (2.10)$$

Now the equation (2.10) is the Volterra criterion of the correct restriction L of the maximal operator \widehat{L} generated by the Laplace operator (1.3) in $L_2(\Omega)$, where Ω is the unit disk.

Further, we apply to the equation (2.10) the Poisson operator of the variables r and φ . The first summand we transform with the formula of the superposition of two Poisson integrals (see [10, p. 140]), and in the second summand the harmonic function $F_j(r, \varphi)$ is reproduced by Poisson's formula. We have

$$\begin{aligned} & \int_t^1 \frac{1 - r^2(1 - \tau^2)^2}{1 - 2r(1 - \tau^2) \cos(\varphi - \gamma) + r^2(1 - \tau^2)^2} \cdot \frac{\tau^2 - t^2}{2\tau} d\tau \\ & + \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 \frac{1 - \left(\frac{\rho^2 - t^2}{\rho}\right)^2}{1 - 2\left(\frac{\rho^2 - t^2}{\rho}\right) \cos(\theta - \gamma) + \left(\frac{\rho^2 - t^2}{\rho}\right)^2} \sum_{j=1}^{\infty} s_j F_j(r, \varphi) \overline{Q_j(\rho, \theta)} \rho d\rho d\theta = 0. \end{aligned}$$

From this equality using the orthonormality of the system $\{F_j(r, \varphi)\}_1^\infty$ we obtain the relation between the orthonormal systems $\{F_j\}_1^\infty$ and $\{Q_j\}_1^\infty$ of the following form

$$\begin{aligned} & \int_t^1 \left\{ \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{1 - r^2(1 - \tau^2)^2}{1 - 2r(1 - \tau^2) \cos(\varphi - \gamma) + r^2(1 - \tau^2)^2} \overline{F_j(r, \varphi)} r dr d\varphi \right\} \cdot \frac{\tau^2 - t^2}{2\tau} d\tau \\ & = -\frac{1}{2\pi} \int_0^{2\pi} \int_t^1 \frac{1 - \left(\frac{\rho^2 - t^2}{\rho}\right)^2}{1 - 2\left(\frac{\rho^2 - t^2}{\rho}\right) \cos(\theta - \gamma) + \left(\frac{\rho^2 - t^2}{\rho}\right)^2} \cdot s_j \overline{Q_j(\rho, \theta)} \rho d\rho d\theta, \quad j = 1, 2, \dots \end{aligned} \quad (2.11)$$

In both parts of the equality (2.11) we use the expansion of the Poisson kernel

$$\frac{1 - r^2}{1 - 2r \cos \varphi + r^2} = 1 + 2 \sum_{n=1}^{\infty} r^n \cos n\varphi.$$

We obtain the equality of the two Fourier series in the orthogonal system $\{1/2, \cos \gamma, \sin \gamma, \dots, \cos n\gamma, \sin n\gamma, \dots\}$ in $L_2(0, 2\pi)$. Equating the coefficients, we get the following system of equations

$$\left\{ \begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \overline{F_j(r, \varphi)} \cdot r dr d\varphi \cdot \int_t^1 \frac{\tau^2 - t^2}{2\tau} d\tau = -\frac{1}{2\pi} \int_0^{2\pi} \int_t^1 s_j \overline{Q_j(\rho, \theta)} \rho d\rho d\theta, \\ & \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \overline{F_j(r, \varphi)} \cdot r^{n+1} \cos n\varphi dr d\varphi \int_t^1 (1 - \tau^2)^n \frac{\tau^2 - t^2}{2\tau} d\tau \\ & \quad = -\int_t^1 \frac{1}{\pi} \int_0^{2\pi} s_j \overline{Q_j(\rho, \theta)} \cdot \cos n\theta d\theta \left(\frac{\rho^2 - t^2}{\rho}\right)^n \rho d\rho, \\ & \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \overline{F_j(r, \varphi)} \cdot r^{n+1} \cdot \sin n\varphi dr d\varphi \int_t^1 (1 - \tau^2)^n \frac{\tau^2 - t^2}{2\tau} d\tau \\ & \quad = -\int_t^1 \frac{1}{\pi} \int_0^{2\pi} s_j \overline{Q_j(\rho, \theta)} \cdot \sin n\theta d\theta \left(\frac{\rho^2 - t^2}{\rho}\right)^n \rho d\rho, \quad j = 1, 2, \dots, \quad n = 1, 2, \dots \end{aligned} \right. \quad (2.12)$$

We denote

$$A_{nj} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \overline{F_j(r, \varphi)} \cos n\varphi \cdot r^n \cdot r dr d\varphi, \quad n = 0, 1, 2, \dots$$

$$B_{nj} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \overline{F_j(r, \varphi)} \sin n\varphi \cdot r^n \cdot r dr d\varphi, \quad n = 1, 2, \dots$$

From the first equation of the system (2.12) it is easy to find that

$$\frac{1}{2\pi} \int_0^{2\pi} s_j \overline{Q_j(t, \theta)} d\theta = \frac{1}{2} A_{0j} \cdot \ln t.$$

The second equation reduces to

$$\int_t^1 (1 - \tau^2)^n \frac{\tau^2 - t^2}{2\tau} d\tau = - \int_t^1 (\rho^2 - t^2)^n \omega_n(\rho) \rho d\rho, \quad n = 1, 2, \dots \quad (2.13)$$

if we denote by

$$\omega_n(\rho) = \frac{1}{\pi} \int_0^{2\pi} s_j \overline{Q_j(\rho, \theta)} \cos n\theta d\theta \frac{1}{A_{nj} \rho^n}.$$

The third equation is transformed into the same equation (2.13), if we denote by

$$\omega_n(\rho) = \frac{1}{\pi} \int_0^{2\pi} s_j \overline{Q_j(\rho, \theta)} \sin n\theta d\theta \frac{1}{B_{nj} \rho^n}.$$

We solve the equation (2.13) with respect to $\omega_n(\rho)$. Note that

$$\omega_1(t) = -\frac{1-t^2}{2t^2}, \quad \omega_2(t) = -\frac{1-t^4}{4t^4};$$

Further, we get the recurrence relation

$$(1-t^2)^{n-k} = - \int_t^1 (\rho^2 - t^2)^{n-k-2} \cdot (n-k)(n-k-1) \cdot 4t^2 \omega_n(\rho) \rho d\rho$$

$$+ k \int_t^1 (\rho^2 - t^2)^{n-k-1} \cdot (n-k) \cdot 4\omega_n(\rho) \rho d\rho, \quad n = 2, 3, 4, \dots, \quad k = 0, 1, 2, \dots, n-2.$$

This relation is equivalent to the Cauchy problem

$$\omega'_n(t) + \frac{2n}{t} \omega_n(t) = \frac{1}{t}, \quad \omega_n(1) = 0.$$

Solving, we get

$$\omega_n(t) = \frac{1-t^{-2n}}{2n}.$$

Now we have the following relations between the orthonormal systems $\{Q_j\}_1^\infty$ and $\{F_j\}_1^\infty$:

$$\begin{cases} \frac{1}{2\pi} \int_0^{2\pi} s_j \overline{Q_j(t, \theta)} d\theta = \frac{1}{2} A_{0j} \cdot \ln t, \\ \frac{1}{\pi} \int_0^{2\pi} s_j \overline{Q_j(t, \theta)} \cdot \cos n\theta d\theta = -A_{nj} \cdot \frac{t^n - t^{-n}}{2n}, \\ \frac{1}{\pi} \int_0^{2\pi} s_j \overline{Q_j(t, \theta)} \cdot \sin n\theta d\theta = -B_{nj} \cdot \frac{t^n - t^{-n}}{2n}, \end{cases} \quad n = 1, 2, \dots \quad (2.14)$$

Satisfying the Volterra criterion (2.10), we obtained the relation (2.14). By assumption $Q_j(t, \theta)$ from $L_2(\Omega)$. Then the integral with respect to t on the left-hand sides of the system of equations (2.14) exists. However, for an arbitrary orthonormal system $\{F_j\}_1^\infty$, for $n = 1, 2, \dots$, the integral on the right-hand sides of the system of equations (2.14) with respect to t from 0 to 1 does not exist. This means that there are no orthonormal systems $\{F_j\}_1^\infty$ and $\{Q_j\}_1^\infty$ satisfying the equality (2.10). This contradicts our assumption that there exists a Volterra correct restriction L . Thus, Theorem 2.1 is proved. \square

Corollary 2.2. *There does not exist a Volterra correct extension L of the minimal operator L_0 generated by the Laplace operator (1.3) in a Hilbert space $L_2(\Omega)$, where Ω is the unit disk.*

Proof. Suppose that there exists a Volterra correct extension L of the minimal operator L_0 . From $L_0 \subset L$ it follows that $L^* \subset L_0^* = \widehat{L}$. The adjoint of a Volterra operator is a Volterra operator. Then we get a contradiction to Theorem 2.1. This completes the proof of Corollary 2.2. \square

In the author's work (see [5]), it was proved that there are no Volterra correct extensions or restrictions for the m -dimensional Laplace operator in $L_2(\Omega)$, where Ω is a bounded domain in \mathbb{R}^m with a sufficiently smooth boundary, if the operator K from representation (1.4) that it belongs to the Schatten class $\mathfrak{S}_p(L_2(\Omega))$ for $0 < p \leq m/2$, where $m \geq 2$.

It was noticed that in the case $m = 1$ there exists many Volterra correct restrictions and extensions.

Remark 2.3. If in (2.11) the function $u(r, \varphi)$ does not depend on the angle φ , then we get one-dimensional equation

$$\widehat{L}u \equiv -\Delta u = -u_{rr} - \frac{1}{r}u_r = f(r), \quad (2.15)$$

in the weighted space $L_2(r; 0, 1)$ with weight r . Then the Volterra criterion (2.10) and the equation (2.14) determine the operator K of the following form

$$Kf = \int_0^1 f(t) \ln t \cdot t dt.$$

To it corresponds to the correct restriction L with domain $D(L) = \{u \in W_2^1(r; 0, 1) : u(0) = 0\}$. Then the correct restriction L is a Volterra, because its inverse operator

$$u(r) = L^{-1}f = \int_0^r \ln \frac{t}{r} f(t) t dt,$$

is a Volterra in space $L_2(r; 0, 1)$.

Remark 2.4. Theorem 2.1 is true for every bounded simply connected domain in the plane, for which the Dirichlet problem is correct and there exists a conformal mapping onto the unit disk.

Remark 2.5. The generalization of Theorem 2.1 to the m -dimensional ball (where $m \geq 3$) does not cause problems but it is cumbersome to write down.

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Bazarkan Nuroldinovich Biyarov
 Department of Fundamental Mathematics
 L.N. Gumilyov Eurasian National University
 2, Satpayev St
 010008 Astana, Kazakhstan
 E-mail: bbiyarov@gmail.com